

CLOSED SUBALGEBRAS OF GROUP ALGEBRAS

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Introduction. Let G be a compact abelian group with character group Γ . For basic definitions the reader is advised to see Chapters 1 and 9 of Rudin [7]. Each closed subalgebra A of the group algebra $L^1(G)$ induces an equivalence relation $\sim$ on Γ : $\alpha \sim \beta$ if and only if $\hat{f}(\alpha) = \hat{f}(\beta)$ every f in A , where $\hat{f}$ is the Fourier transform of f . We will denote the equivalence classes of $\sim$ by $\{E_\lambda\}$ where λ is in some index set and distinguish one special class $E_0 = \{\gamma \in \Gamma : \hat{f}(\gamma) = 0 \text{ for every } f \text{ in } A\}$ called the zero set of A . Since Γ is discrete and $\hat{f} \in C_0(\Gamma)$ for every f in $L^1(G)$, each E_λ , $\lambda \neq 0$, is finite. E_0 may be infinite.

For $\lambda \neq 0$ let P_λ be the trigonometric polynomial whose transform is the characteristic function of E_λ , and A_0 the smallest closed algebra containing every P_λ . Rudin [7, p. 231] has shown that A_0 induces $\sim$ and is contained in every closed subalgebra which induces $\sim$ with zero set E_0 . If we define A^0 to be the algebra of all functions whose transforms are constant on every E_λ and zero on E_0 then A^0 will be the largest closed algebra inducing $\sim$.

Rudin [7] asked if there exist distinct closed subalgebras which induce the same equivalence relation. Or, equivalently, does there exist a closed subalgebra A with $A^0 \neq A_0$? Kahane [4] gives a negative answer for $G = T$, the circle group.

In §1 we give sufficient conditions for an equivalence relation on the integers Z to be uniquely induced by exactly one closed subalgebra of $L^1(T)$. This result strengthens Theorem 3 of Kahane [4]. We also prove his result on $Z \times Z$.

In §2 we study the algebras A_0 and A^0 in detail and give necessary and sufficient conditions for $A_0 = A^0$.

Finally in §3 we will consider factorization in closed subalgebras of $L^1(G)$ and show that although $L^1(G)$ has factorization, there exists a closed subalgebra A without factorization and such that $A \cap L^p(G)$ is L^1 -dense in A , where $1 < p < 2$.

0. Notation. If B is a linear subspace of $L^1(G)$ we define its annihilator $B^\perp = \{\phi \in L^\infty(G) : \phi(f) = 0 \text{ for every } f \text{ in } B\}$ where $\phi(f) = \int_G f(-x)\phi(x) dx$. (dx denotes normalized Haar measure.)

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Since G is compact we have $L^\infty(G) \subset L^1(G)$, and it makes sense to consider the Fourier transform of $\phi \in L^\infty(G)$, defined $\hat{\phi}(\gamma) = \int_G \phi(x) \gamma(-x) dx$ where $\gamma \in \Gamma$. Hence, for a trigonometric polynomial $f = \sum_{i=1}^n a_i \gamma_i$, $\gamma_i \in \Gamma$ we have

$$\begin{aligned} \phi(f) &= \int_G \left(\sum_{i=1}^n a_i \gamma_i(-x) \right) \phi(x) dx \\ &= \sum_{i=1}^n a_i \int_G \gamma_i(-x) \phi(x) dx = \sum_{i=1}^n a_i \hat{\phi}(\gamma_i). \end{aligned}$$

For f and g in $L^1(G)$ we define convolution $(f * g)(x) = \int_G f(x-y)g(y) dy$. Thus, with the above notation, it follows that $\phi(f) = (\phi * f)(0)$.

I. Synthesis. In this section we will be concerned specifically with the circle group T and the torus T^2 . Z will denote the group of integers. We will exhibit a class of partitions of Z and $Z \times Z$ which are induced by exactly one closed subalgebra of $L^1(T)$ and $L^1(T^2)$, respectively.

First, we will generalize Theorem 2 of Kahane [4] on Z and second, we will prove an analogous result on $Z \times Z$.

For the following definition G will be an arbitrary compact abelian group and Γ its dual group.

1.1. DEFINITION. An equivalence relation $\sim$ on Γ with a distinguished class E_0 is said to have synthesis if it is induced by exactly one closed subalgebra of $L^1(G)$ with zero set E_0 .

Kahane [4] has shown that if $\sim$ is an equivalence relation on Z which satisfies: $n \sim m$ implies $|n - m| \leq M$, where M is a fixed constant, then $\sim$ has synthesis.

The following is a generalization of Kahane's result.

1.2. THEOREM. Let $\sim$ be an equivalence relation on Z with a distinguished class E_0 . Suppose that for $n, m \notin E_0$, $n \sim m$ and $|n| \leq |m|$ we have that $|m - n| \leq M|n|^{1/2}$ (M a constant). Then $\sim$ has synthesis.

Proof. It is well known that if $h \in L^1(T)$ and is continuous at $x_0 \in T$, then the Cesaro sums $\{U_n\}$ of its Fourier series $\sum_{j=-\infty}^{\infty} \hat{h}(j)e^{ijx}$ converge to h at the point x_0 . By a standard computation we may write

$$(1) \quad U_n(x) = \sum_{j=-\infty}^{\infty} \hat{K}_n(j) \hat{h}(j) e^{ijx}$$

where $\hat{K}_n(j) = \max(0; 1 - |j|/(n+1))$. Now let $\sim$ satisfy the above hypotheses. Also, let A^0 and A_0 be the maximal and minimal subalgebras of $L^1(T)$, respectively, which induce $\sim$. We want to show $A^0 = A_0$ or equivalently, $A^{0\perp} = A_0^\perp$.

Obviously, $A^{0\perp} \subset A_0^\perp$ so we need only prove the reverse containment. For this purpose, choose $f \in A^0$ and $\phi \in A_0^\perp$. We will show $\phi(f) = 0$.

Since $\phi \in L^\infty(G)$ and $f \in L^1(G)$ we have that $\phi * f$ is continuous. In particular, it is continuous at 0. We shall approximate $(\phi * f)(0) = \phi(f)$ by the Cesaro sums

$U_n(0)$ of (1) where $h = \phi * f$. In fact, we will show $U_n(0) \rightarrow 0$ as $n \rightarrow \infty$. Since $\phi \in A_0^\perp$ it follows that $\phi(P_\lambda) = 0$ or

$$(2) \quad \sum_{j \in E_\lambda} \hat{\phi}(j) = 0.$$

Define $j_\lambda = \min \{|j|; j \in E_\lambda\}$. Now using (2) and the fact that $\hat{f}$ is constant on every E_λ , we have

$$(3) \quad U_n(0) = \sum_{\lambda \neq 0} \sum_{j \in E_\lambda} \{\hat{K}_n(j) - \hat{K}_n(j_\lambda)\} \hat{f}(j) \hat{\phi}(j)$$

where the inner sum extends over those λ for which $j_\lambda \leq n$. If $j \in E_\lambda$ for such a λ we have $|j - j_\lambda| \leq Mj_\lambda^{1/2}$ or $|j| \leq j_\lambda + Mj_\lambda^{1/2} \leq (M+1)j_\lambda \leq (M+1)n$. Hence, it follows that $\hat{K}_n(j_\lambda) - \hat{K}_n(j) \leq (M+1)/n^{1/2}$. Thus, setting $N = (M+1)n$ and applying the Cauchy-Schwarz inequality and the Plancherel theorem to (3) we obtain

$$|U_n(0)| \leq \frac{(M+1)}{n^{1/2}} \sum_{j=-N}^N |\hat{f}(j) \hat{\phi}(j)| \leq (M+1) \left[\frac{1}{N} \sum_{j=-N}^N |\hat{f}(j)|^2 \right]^{1/2} \|\phi\|_2$$

which tends to 0 as $N \rightarrow \infty$ since $\phi \in L^2(G)$ and $\hat{f} \in C_0(\Gamma)$.

Daniel Rider [6] has an example of an equivalence relation $\sim$ on Z satisfying: $n \sim m$, $|n| \leq |m|$ implies $|n - m| \leq |n|$, and does not have synthesis.

This yields the following interesting question which we have not been able to answer: Suppose $\sim$ is an equivalence relation on Z satisfying: $n \sim m$, $|n| \leq |m|$ implies $|n - m| \leq M|n|^\delta$, where M is some constant and $1/2 < \delta < 1$. For which values of δ does $\sim$ have synthesis?

The more ambitious task of classifying all synthetic partitions is beyond our reach. However, suppose S is a "Sidon set" (see [7, pp. 121-130] for the definition and facts about Sidon sets) and A is a closed subalgebra of $L^1(T)$ with $Z - E_0 \subset S$. Then $A^0 = A_0$. In fact, it will follow that if B is a closed subalgebra of A then $B^0 = B_0$.

The problem of discovering synthetic equivalence relations on $Z \times Z$ is open. Although we will prove an extension of Kahane's result to $Z \times Z$, the correspondingly stronger result of (1.2) has eluded us.

We begin with some facts and definitions which may all be found in Zygmund [8, Vol. II].

The torus will be denoted T^2 . Let $R(n) = \{(i, j) \in Z \times Z : |i| \leq n, |j| \leq n\}$. For $h \in L^1(T^2)$ let

$$(1) \quad S_n(\bar{x}) = \sum_{j \in R(n)} \hat{h}(j) e^{i(j \cdot \bar{x})}$$

where $\bar{x} \in T^2$, and $\cdot$ represents the usual dot product.

By a standard manipulation we may write the Cesaro sums $\{U_n(\bar{x})\}$ of (1) by

$$(2) \quad U_n(\bar{x}) = \frac{1}{(n+1)^2} \sum_{j \in R(n)} (n - |j_1| + 1)(n - |j_2| + 1) \hat{h}(j) e^{i(j \cdot \bar{x})}$$

where $j = (j_1, j_2)$.

1.3. THEOREM [8, VOL. II, P. 304]. Suppose $h \in L^\infty(T^2)$ and h is continuous at $\bar{x}_0 \in T^2$. Then $|h(\bar{x}_0) - U_n(\bar{x}_0)| \rightarrow 0$ as $n \rightarrow \infty$.

1.4. THEOREM. Let $\sim$ be an equivalence relation on $Z \times Z$ which satisfies; $(n_1, m_1) \sim (n_2, m_2)$ implies $|n_1 - n_2| \leq M$ and $|m_1 - m_2| \leq M$ for $(n_i, m_i) \notin E_0$, $i = 1, 2$, where M is some constant. Then $\sim$ has synthesis.

Proof. The proof is similar to that of (1.2). We define

$$\hat{K}_n(i, j) = \max(0; 1 - |i|/(n+1)) \max(0; 1 - |j|/(n+1))$$

$i_\lambda = \min(|i|; (i, j) \in E_\lambda)$, and $j_\lambda = \min(|j|; (i, j) \in E_\lambda)$. Now if $\phi \in A_0^\perp$, $f \in A^0$ we let $h = \phi * f$, $\bar{x}_0 = 0$ and rewrite (2) as

$$\begin{aligned} U_n(0) &= \sum_{(i,j) \in Z \times Z} \hat{K}_n(i, j) \hat{\phi}(i, j) \hat{f}(i, j), \\ &= \sum_{\lambda \neq 0} \sum_{(i,j) \in E_\lambda} \{\hat{K}_n(i, j) - \hat{K}_n(i_\lambda, j_\lambda)\} \hat{f}(i, j) \hat{\phi}(i, j) \end{aligned}$$

where the inner sum extends over those λ for which $i_\lambda \leq n$ and $j_\lambda \leq n$. If $(i, j) \in E_\lambda$ for such a λ then $|i| \leq (M+1)n$ and $|j| \leq (M+1)n$. Hence $\hat{K}_n(i_\lambda, j_\lambda) - \hat{K}_n(i, j) \leq C/n$ where C is a constant independent of n . Thus, as in the proof of (1.2) we have

$$|U_n(0)| \leq \frac{C}{n} \left[\sum_{(i,j) \in E(n)} |f(i, j)|^2 \right]^{1/2} \|\phi\|_2$$

which tends to 0 as $n \rightarrow \infty$.

II. **The algebras A_0 and A^0 .** Throughout this section, G will be an arbitrary compact abelian group and A a closed subalgebra of $L^1(G)$ with induced equivalence relation $\sim$ and zero set E_0 . We will be concerned with finding conditions which will guarantee $A_0 = A^0$. It will be shown in (2.1) that although A_0 and A^0 have the same maximal ideal space, their annihilators $A_0^\perp$ and $A^{0\perp}$ respectively, can be characterized in (2.3) quite simply. We will construct a linear subspace B of $L^\infty(G)$ consisting of trigonometric polynomials such that $A^{0\perp} = w^*$ -closure of B and $A_0^\perp = (L^2\text{-closure of } B) \cap L^\infty(G)$.

Denote the maximal ideal space of A by $\Delta(A)$ and let Γ' be the quotient space obtained from $\Gamma \setminus E_0$ by identifying equivalent elements of Γ . If we let γ' be the image of γ under the quotient map, it is clear that γ' induces a nonzero homomorphism h on A defined by $h(f) = \hat{f}(\gamma')$ for $f \in A$. However, the converse is also true.

2.1. THEOREM. Every nonzero homomorphism on A is induced by an element of Γ' .

Proof. Define $T: A \rightarrow C_0(\Gamma')$ by $T(f)(\gamma') = \hat{f}(\gamma')$. It is easy to show that $T(A_0)$ is a separating, conjugate-closed subalgebra of $C_0(\Gamma')$. Since $A_0 \subset A$ (Rudin [7, p. 232]) we have by the Stone-Weierstrass theorem that $T(A)$ is dense in $C_0(\Gamma')$ under the uniform norm. It follows that the adjoint map $T^*: \Gamma' \rightarrow \Delta(A)$ carries

Γ' homeomorphically onto a closed subset of $\Delta(A)$ (see for example [5, p. 76]). It remains to show $T^*(\Gamma') = \Delta(A)$. Choose $h \in \Delta(A)$. Then for $f \in A$

$$|h(f)|^n = |h(f^n)| \leq \|f^n\|_A = \|f^n\|_{L^1(G)}.$$

By the spectral radius formula we obtain

$$|h(f)| \leq \|f\|_{C_0(\Gamma')} = \|\hat{f}\|_{C_0(\Gamma')}.$$

Hence, the function $h'(T(f)) = h(f)$ is a continuous homomorphism on $T(A)$ and may be extended to all of $C_0(\Gamma')$ since $T(A)$ is dense in $C_0(\Gamma')$. Thus, there exists $\gamma' \in \Gamma'$ with $h(f) = T^*(\gamma')f$ for every $f \in A$. Consequently, T^* is onto.

The above theorem suggests that Banach algebra methods of distinguishing A_0 and A^0 will be of little help. Also, the fact that A_0 is generated as a vector space as well as an algebra by the P_λ leads us to study the annihilators $A_0^\perp$ and $A^{0\perp}$. For this purpose we introduce the following notation.

Let $\sim$ have equivalence classes $\{E_\lambda\}$. For $\lambda \neq 0$ and $E_\lambda = \{\gamma_1, \dots, \gamma_n\}$ define

$$B_\lambda = \left\{ \phi \in L^\infty(G) : \phi = \sum_{i=1}^n a_i \gamma_i \text{ where } a_1, \dots, a_n \text{ are complex numbers with } \sum_{i=1}^n a_i = 0 \right\}$$

and let B_0 be the linear space of $L^\infty(G)$ generated by the elements of E_0 . (If $E_0 = \emptyset$, let $B_0 = \emptyset$.) Finally, let B be the linear space of $L^\infty(G)$ generated by the elements of $\bigcup B_\lambda$.

2.2. LEMMA. *For any closed subalgebra A of $L^1(G)$, we have $B \subset A^\perp$.*

Proof. Pick $f \in A$ and an index λ . If $\lambda \neq 0$ and $\phi \in B_\lambda$, then $\phi = \sum_{i=1}^n a_i \gamma_i$ with $\sum_{i=1}^n a_i = 0$ where $E_\lambda = \{\gamma_1, \dots, \gamma_n\}$. Then

$$\phi(f) = \int f(x) \phi(-x) dx = \int f(x) \sum_{i=1}^n a_i \gamma_i(-x) dx = \sum_{i=1}^n a_i \hat{f}(\gamma_i) = \hat{f}(E_\lambda) \sum_{i=1}^n a_i = 0.$$

If $\lambda = 0$ and $\phi \in B_0$, then $\phi = \sum_{i=1}^k b_i \beta_i$ with $\beta_i \in E_0$ for $i = 1, \dots, k$. Then, similarly, $\phi(f) = \sum_{i=1}^k b_i \hat{f}(\beta_i) = 0$ since $\hat{f}(\beta_i) = 0$ for $i = 1, \dots, k$. Since the elements of the B_λ generate B and are contained in the linear space $A^\perp$, we have $B \subset A^\perp$.

2.3. THEOREM. *Let A be a closed subalgebra of $L^1(G)$, then*

- (1) $A_0^\perp = (L^2\text{-closure of } B) \cap L^\infty(G)$
- (2) $A^{0\perp} = w^*\text{-closure of } B$.

Proof. (i) By (2.2) we have $A^\perp \supset B$. Now choose $\phi \in (L^2\text{-closure of } B) \cap L^\infty(G)$. We will show $\phi \in A_0^\perp$.

Now, there exists a sequence $\phi_1, \phi_2, \dots$ in B with $\|\phi - \phi_n\|_2 \rightarrow 0$ as $n \rightarrow \infty$. Suppose $P_\lambda \in A_0$. Applying the Cauchy-Schwarz inequality and using the fact that $\phi_n(P_\lambda) = 0$ for every n , we have

$$\begin{aligned} |\phi(P_\lambda)| &= \left| \int \phi(x) P_\lambda(-x) dx \right| \leq \left| \int (\phi(x) - \phi_n(x)) P_\lambda(-x) dx \right| + |\phi_n(P_\lambda)| \\ &\leq \|\phi - \phi_n\|_2 \cdot \|P_\lambda\|_2 \rightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Thus, $\phi(P_\lambda) = 0$. Consequently, since linear combinations of the P_λ are dense in A_0 , we have $\phi \in A_0^\perp$. This yields

$$(L^2\text{-closure of } B) \cap L^\infty(G) \subset A_0^\perp.$$

(ii) Choose $\phi \in A_0^\perp$. Clearly $\phi \in L^\infty(G)$. We will show $\phi \in (L^2\text{-closure of } B)$. Let $\lambda_1, \lambda_2, \dots$ be the indices for which $\hat{\phi}(E_{\lambda_i}) \neq \{0\}$ and $\{\beta_1, \beta_2, \dots\} = E_0 \cap \text{support of } \hat{\phi}$. If $E_\lambda = \{\gamma_1, \dots, \gamma_n\}$ then $0 = \phi(P_\lambda) = \sum_{i=1}^n \hat{\phi}(\gamma_i)$. So $\phi_\lambda = \sum_{i=1}^n \hat{\phi}(\gamma_i) \gamma_i \in B_\lambda \subset B$. By the Plancherel theorem ϕ is the L^2 -limit of the sums $\sum_{i=1}^k (\phi_{\lambda_i} + \beta_i) \in B$. Hence, $\phi \in (L^2\text{-closure of } B) \cap L^\infty(G)$. We have proven

$$A_0^\perp \subset (L^2\text{-closure of } B) \cap L^\infty(G).$$

Consequently, (i) and (ii) yield (1).

(iii) Using (2.2) we have $A^{0\perp} \supset B$. But every annihilator in $L^\infty(G)$ is w^* -closed. Thus,

$$A^{0\perp} \supset w^*\text{-closure of } B.$$

(iv) We will show $A^{0\perp} \subset w^*\text{-closure of } B$. Let $D = \{f \in L^1(G) : \phi(f) = 0 \text{ for every } \phi \in B\}$. Then $D^\perp = w^*\text{-closure of } B$. Hence, it will suffice to show $A^0 \supset D$, for then $A^{0\perp} \subset D^\perp$.

Pick $g \in L^1(G)$ with $g \notin A^0$. Then either there exists $\gamma_1 \sim \gamma_2$ with $\gamma_1, \gamma_2 \notin E_0$ and $\hat{g}(\gamma_1) \neq \hat{g}(\gamma_2)$ or there exists $\beta \in E_0$ with $\hat{g}(\beta) \neq 0$. If the former is true, let $\Phi = \gamma_1 - \gamma_2 \in B$ then $\Phi(g) \neq 0$ and hence $g \notin D$. Otherwise, let $\Psi = \beta \in B$ and conclude $\Psi(g) \neq 0$. Consequently $A^0 \supset D$, or

$$A^{0\perp} \subset w^*\text{-closure of } B.$$

Combining (iii) and (iv), we have proven (2).

III. Factorization. If a commutative Banach algebra B has an approximate identity there exists a bounded set U of B such that x belongs to the closure of xU for every x in B . It is this property that motivates the following definition.

3.1. DEFINITION. Let B be a commutative Banach algebra. We say that B has a *pseudo-identity* (p.i.) if and only if x belongs to the closure of xB for every x in B .

It is well known that $L^1(G)$ has an approximate identity which may be chosen to be a (p.i.) for $L^p(G)$, $1 \leq p < \infty$. But if $1 < p \leq \infty$, it will follow from the Hausdorff-Young theorem that $L^p(G)$ has no approximate identity.

Cohen [1] has shown that any Banach algebra B with approximate identity has factorization, i.e., $B = B \cdot B$. In particular, $L^1(G) = L^1(G) * L^1(G)$. One may ask if this property is transferred to all the closed subalgebras of $L^1(G)$. We will use a result of Daniel Rider [6] to prove the existence of a closed subalgebra A without factorization and $A \cap L^p(G)$ is L^1 -dense in A for $1 < p < 2$.

We begin with a definition.

3.2. DEFINITION. Let $1 < p \leq \infty$. A closed subalgebra A of $L^1(G)$ is a p -algebra if and only if $A \cap L^p(G)$ is L^1 -dense in A .

Clearly, A_0 is a p -algebra for every p and every algebra A .

For what follows, we will adopt the notation: $A^2 = A * A$ and $A^j = A^{j-1} * A$ for $j = 3, 4, \dots$, where A is a closed subalgebra of $L^1(G)$.

3.3. LEMMA. Let A be a p -algebra. Then there exists an integer k such that $A^m \subset A_0$ for $m \geq k$.

Proof. Case (i). Assume $p \geq 2$. Then $A_0 \subset A = L^1$ -closure of $(A \cap L^p(G)) \subset L^1$ -closure of $(A \cap L^2(G)) \subset A_0$. Hence, $A = A_0$ and so $A^m \subset A_0$ for every integer m .

Case (ii). Assume $1 < p < 2$. Choose k such that $p^{(k-1)} > q$ where $p^{-1} + q^{-1} = 1$. Now suppose $m \geq k$ and $f = f_1 * \dots * f_m$ where $f_j \in A$ for $j = 1, 2, \dots, m$. We will show $f \in A_0$. Let $\varepsilon > 0$ be given.

Since A is a p -algebra, we may choose $h = h_1 * \dots * h_m$ with $h_j \in A \cap L^p(G)$ for $j = 1, 2, \dots, m$ and $\|h - f\|_1 < \varepsilon$. If we define $\phi_j = \hat{h}_j$ for $j = 1, 2, \dots, m$ it will follow from the Hausdorff-Young theorem that $\phi_j \in L^q(\Gamma)$ for each j .

Let K be any finite subset of Γ . Then by the Hölder inequality we have

$$(1) \quad \sum_{\gamma \in K} |\hat{h}(\gamma)| = \sum_{\gamma \in K} |\phi_1 \cdots \phi_m(\gamma)| \leq \left[\sum_{\gamma \in K} |\phi_1|^q(\gamma) \right]^{q^{-1}} \left[\sum_{\gamma \in K} |\phi_2 \cdots \phi_m|^p(\gamma) \right]^{p^{-1}} \\ \leq \|\phi_1\|_q \left[\sum_{\gamma \in K} |\phi_2 \cdots \phi_m|^p(\gamma) \right]^{p^{-1}}.$$

By repeated applications of the Hölder inequality to (1) and the fact that $p^{(m-1)} \geq p^{(k-1)} > q$ we obtain

$$(2) \quad \sum_{\gamma \in K} |\hat{h}(\gamma)| \leq C \left[\sum_{\gamma \in K} |\phi_m(\gamma)|^{p^{(m-1)}} \right]^{p^{-(m-1)}} \\ \leq C \left[\sum_{\gamma \in K} |\phi_m(\gamma)|^q \right]^{p^{-(m-1)}} \leq C \|\phi_m\|_q$$

where C is a constant independent of K . Since K was arbitrary, it follows that $h \in L^1(\Gamma) \subset L^2(\Gamma)$. Hence by the Plancherel theorem we have $h \in (L^2(G) \cap A) \subset A_0$. Since ε was arbitrary, we conclude $f \in A_0$.

3.4. THEOREM. If A is a p -algebra such that $A \subset L^1$ -closure of A^2 then $A = A_0$.

Proof. From (3.3) we may pick k such that $A^m \subset A_0$ for $m \geq k$. It follows that $A_0 \subset A \subset L^1$ -closure of $A^2 \subset \cdots \subset L^1$ -closure of A^{2^t} where t is chosen to satisfy $2^t > k$. Hence $A = A_0$.

The following corollaries are immediate.

3.5. COROLLARY. *If A is a p -algebra with (p.i.) then $A = A_0$.*

3.6. COROLLARY. *If A is a p -algebra with factorization then $A = A_0$.*

Rider [6] has constructed a closed subalgebra A of $L^1(T)$ which is generated by a function $f \in L^p(T)$ ($1 < p < 2$) such that $A^0 = A \neq A_0$. It follows from (3.5) that A does not have a (p.i.) and from (3.6) that A does not have factorization.

Now let G be a locally compact abelian group and I a closed ideal of $L^1(G)$. Define $Z(I) = \{\gamma \in \Gamma : \hat{f}(\gamma) = 0 \text{ for every } f \in I\}$. The problem of "spectral synthesis" is concerned with the following question: Do there exist two distinct closed ideals I_1 and I_2 of $L^1(G)$ such that $Z(I_1) = Z(I_2)$? Malliavin [7, p. 172] has shown that this will always be the case if G is not compact. For G compact, $Z(I) = E_0$, and it is clear that $Z(I_1) = Z(I_2)$ implies $I_1 = I_2$.

Helson [3] has shown that if I_1 and I_2 are two distinct closed ideals of $L^1(G)$, G not compact, with $I_1 \subset I_2$ and $Z(I_1) = Z(I_2)$ then there exists a closed ideal I with $Z(I) = Z(I_1) = Z(I_2)$ and $I_1 \subsetneq I \subsetneq I_2$. One is struck with the following question for G compact: Suppose A is a closed subalgebra of $L^1(G)$ with $A \neq A_0$. Does there exist a closed subalgebra A_1 with $A \supsetneq A_1 \supsetneq A_0$? We will use Rider's example and (3.3) to show that this is not generally the case.

For this purpose, let A be a closed subalgebra of $L^1(T)$ with $A \neq A_0$ and generator $f \in L^p(G)$, $1 < p < 2$. Now, functions of the form $h(f)$ where h is a polynomial with $h(0) = 0$ are dense in A . If $\tilde{f}$ represents the image of f under the quotient map $A \rightarrow A/A_0$, we have that the functions $h(\tilde{f})$ are dense in the quotient algebra A/A_0 . Now by (3.3) there exists a positive integer k such that $f^n \in A_0$ for $n \geq k$. Since every finite dimensional linear subspace of A/A_0 is closed, it follows that A/A_0 is a finite dimensional vector space with generators $\tilde{f}, \tilde{f}^2, \dots, \tilde{f}^{(k-1)}$ and thus has dimension less than k .

Suppose we could always find a closed subalgebra B_1 to satisfy $B \supsetneq B_1 \supsetneq B_0$ whenever B was a closed subalgebra with $B \neq B_0$. Then by induction we may choose k distinct closed subalgebras of $L^1(G)$ say $A_1, A_2, \dots, A_k$ with $A \supsetneq A_1 \supsetneq \cdots \supsetneq A_k \supsetneq A_0$. But then $A/A_0 \supsetneq A_1/A_0 \supsetneq \cdots \supsetneq A_k/A_0 \supsetneq \{0\}$ implies A/A_0 has dimension greater than k . Contradiction.

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